A Quantum Algorithm for Minimum Steiner Tree Problem

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Abstract. Minimum Steiner tree problem is a well-known NP-hard problem. For the minimum Steiner tree problem in graphs with n vertices and k terminals, there are many classical algorithms that take exponential time in k. In this paper, to the best of our knowledge, we propose the first quantum algorithm for the minimum Steiner tree problem. The complexity of our algorithm is $\mathcal{O}^*(1.812^k)$. A key to realize the proposed method is how to reduce the computational time of dynamic programming by using a quantum algorithm because existing classical (non-quantum) algorithms in the problem rely on dynamic programming. Fortunately, dynamic programming is realized by a quantum algorithm for the travelling salesman problem, in which Grover's quantum search algorithm is introduced. However, due to difference between their problem and our problem to be solved, recursions are different. Hence, we cannot apply their technique to the minimum Steiner tree problem in that shape. We solve this issue by introducing a decomposition of a graph proposed by Fuchs *et al.*

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1 Introduction

Given an undirected graph $G = (V, E)$, a weight $w : E \to \mathbb{R}^+$, and a subset of vertices $K \subseteq V$, usually referred to as terminals, a Steiner tree is a tree that connects all vertices in K. In this paper, let $n = |V|$ be the size of vertices and $k = |K|$ be the size of terminals. A Steiner tree T is the minimum Steiner tree (MST) when the total edge weight $\sum_{e \in E(T)} w(e)$ is the minimum among all Steiner trees of K. Note that all leaves of a Steiner tree T are vertices in K. The task that finds a minimum Steiner tree is called minimum Steiner tree problem, and this problem is known as an NP-hard problem [1]. Note that for fixed k, this problem can be solved in polynomial time, which means that the minimum Steiner problem is *fixed parameter tractable* [2, 3].Although there are difficulties in solving the minimum Steiner tree problem, this problem is applied to solve problems such as power supply network, communication network and facility location problem [4]. Since these practical problems need to be solved, researching the exponential algorithm that has better base is significant.

A naive way to solve the minimum Steiner tree problem is to compute all possible trees. However, the number of all trees in the graph $G = (V, E)$ is $\mathcal{O}(2^{|E|})$ at worst. However, an exhaustive search is not realistic. *The Dreyfus-Wagner algorithm* (the D-W algorithm) is a well-known algorithm based on dynamic programming for solving the Steiner problem in time $\mathcal{O}^*(3^k)$ [5]. The $\mathcal{O}^*$ notation hides a polynomial factor in n and k. This algorithm has been the fastest algorithm for decades. In 2007, Fuchs *et al.* [6] have improved this to $\mathcal{O}^*(2.684^k)$ and Mölle *et al.* [7] to $\mathcal{O}((2 + \delta)^k n^{f(\delta^{-1})})$ for any constant $\delta > 0$. For a graph with a restricted weight range, Björklund *et al.* have proposed an $\mathcal{O}^*(2^k)$ algorithm using subset convolution and Möbius inversion [8]. An important thing is that the dynamic programming part of these algorithms [7, 6, 8] use the D-W algorithm.

In order to speed up classical algorithms, use of quantum algorithms is an effective technique. In particular, Grover's quantum search (Grover search) [9] and its generalization, quantum amplitude amplification [10, 11], are widely applicable. Grover search brings quadratic speed up to an unstructured search problem [9, 12]. This is one of the advantages quantum algorithms have over classical algorithms. For NP-hard problems, speeding up using Grover search [9] is a typical method. However, simply applying Grover search to a classical algorithm does not always make faster than the best classical algorithm in many problems. For example, in [13], by using quantum computers, the Travelling Salesman Problem (TSP) for a graph which has n vertices is solved in time $\mathcal{O}^*(\sqrt{n!})$ which is the square root of the classical complexity $\mathcal{O}^*(n!)$ of an exhaustive search. However, the best classical algorithm for TSP takes only $\mathcal{O}^*(2^n)$ [14, 15] which is clearly faster than $\mathcal{O}^*(\sqrt{n!})$.

In order to speed up algorithms for the minimum Steiner tree problem, it is thought that use of Grover search is also an effective technique. Combining classical algorithms with Grover search is one of the ways to make an algorithm faster than the best classical algorithm. For example, Ambainis *et al.* [16] have combined Grover search with algorithms for TSP, Minimum Set Cover Problem and so on that use dynamic programming. A naive way is replacing the dynamic programming part of the algorithm of Ambainis *et al.* by D-W algorithm. However, we cannot use the method of Ambainis *et al.* in the same way because the characteris-

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Table 1: Comparison of the algorithms.

Algorithm	Complexity	classical or quantum
Dreyfus and Wagner [5]	$\mathcal{O}^*(3^k)$	classical
Fuchs [6]	$\mathcal{O}^*(2.684^k)$	classical
Mölle [7]	$\mathcal{O}((2+\delta)^k n^{f(\delta^{-1})})$	classical
Björklund [8] (best known in the restricted weight case)	$\mathcal{O}^*(2^k)$	classical
This paper	$\mathcal{O}^*(1.812^k)$	quantum

tic of minimum Steiner tree problem differs from that of TSP. Hence, we adapt this method to a method proposed by Fuchs *et al.* [6] for applying Grover search. The decomposition method of Fuchs *et al.* is optimized for a classical computer. We optimize the decomposition for a quantum computer. Our algorithm achieved the complexity $\mathcal{O}^*(1.812^k)$. This improvement of complexity brings more than 10^4 times speed up compared to $\mathcal{O}^*(2^k)$ even in the case of $k = 100$. Table 1 shows the complexity of classical algorithms for minimum Steiner tree problem and our proposed algorithm.

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